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THE DRIFT VELOCITY OF TRAPPED PARTICLES

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Abstract 15424

The longitudinal drift velocity of a particle trapped in the earth's field is considered. It is shown that there is a relation between drift velocities at different points on any line of force and that the drift velocities of particles with differing mirror points are also related. These relations can be used to simplify the calculation of the average effects of the atmosphere.

author

Introduction

Newkirk and Walt (1964) have shown that the longitudinal drift velocity, of trapped particles moving in the earth's field, varies appreciably with latitude and longitude. The calculation of drift velocity is important in calculating the effects of the atmosphere on the particles. In this paper it is shown that the drift velocity of a particle with a given mirror point, is related in a simple way to the drift velocity at the equator. The drift velocities at the equator for particles with different mirror points, are also related; the relation is given in terms of known functions. These results considerably simplify the integrations which are necessary to compute average effects of the atmosphere.

Ray (1963), Northrop and Teller (1960) and others, have made use of a coordinate representation (α, β) in which $\bar{B} = \bar{\nabla} \alpha \times \bar{\nabla} \beta$. It is shown that the drift velocity results imply a convenient representation of α and β. If α is taken to be J, where J is the usual integral invariant, then β is related to the time it takes for a particle to drift in longitude.

Drift Velocity Calculations for Fixed Bn

A formula for the instantaneous drift velocity has been given by various authors. Lew (1961) shows that in a curl free field

$$\bar{v}_d = \frac{m \gamma c \varkappa}{eB} \left(1/2\ v_\perp^2 + v_\parallel^2\right) \bar{b} \qquad (1)$$

where $m\gamma$ is the mass of the particle and e is its charge: $v_{\perp}$ and $v_{/\!/}$ are the components of v (the velocity) perpendicular to and parallel to the line of force: $\bar{b}$ is a unit vector along the binormal to the magnetic field line. We will use the convention that if $\bar{x}$ is any vector then x

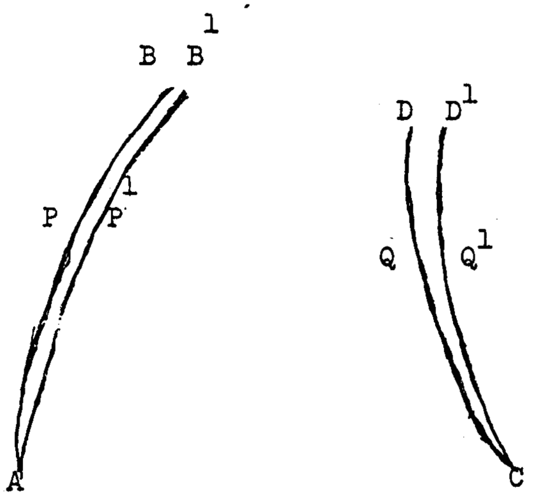

Figure 1. Two neighboring lines of force AB, CD and the associated particle paths AB' and CD'

denotes the magnitude of $\bar{x}$. Northrop and Teller (1960) show that equation (1) can be integrated over a latitudinal bounce to give

$$\bar{u}_d = \frac{c}{eB^2T} \bar{\nabla} J \times \bar{B} \tag{2}$$

where $\bar{u}_d$ is $\bar{v}_d$ averaged over one bounce, and T is the bounce period.

J is the integral invariant

$$J = \int p_{/\!/} \, ds \qquad (3)$$

where ds is an arc length along the line of force, and $p_{/\!/}$ is the component of the momentum along the line of force. Newkirk and Walt (1964) have used formula (2) to compute $\bar{u}_d$ at various longitudes for various values of B_m, the mirror point B.

It has been pointed out by McIlwain (private communication) that there is a relation between $\bar{u}_d$ at various points on the line of force. Suppose the particle is at point A at time zero. AB denotes the line of force through A (See Figure 1). The particle drifts away from the line of force, let AB' denote the actual path of the particle: the difference between AB and AB' represents the effects of instantaneous drift. After several bounce periods let the particle reach point C, by definition

$$\bar{u}_d\,(A) = AC/T\,(AC) \qquad (4)$$

where T (AC) denotes the time to go from A to C. Let CD denote the line of force through C and let CD' be the path of the particle. Consider the same particle at different a point on its path. Let P' be any point on AB' and let Q' be the corresponding point (the point with the same B value) on CD', then

$$u_d\,(P') = P'Q' \Big/ \Big(T(AC) - T(AP') + T(CQ') \Big) \qquad (5)$$

We now make two approximations, namely

$$PQ = P'Q' \tag{6a}$$

and

$$T(AC) = T(AC) - T(AP') + T(CQ') \tag{6b}$$

From equations (4) through (6)

$$\frac{u_d(A)}{u_d(P)} = \frac{AC}{PQ} \tag{7}$$

that is, the drift velocity at any point is proportional to the spread of the lines of force. This is the main result of this section. We have made two approximations, (6a) and (6b), but in addition T(AC) should be short enough so that u_d does not change during the period T(AC). Now the drift period is much longer than the bounce period, hence we can make T(AC) >> T(AB) without any loss of accuracy, thus (6a) and (6b) are accurate to second order.

To illustrate these results consider a particular line of force; in any results relating to the earth's field we have used the Jensen and Cain (1962) 48 term spherical harmonic expansion. The geographic coordinates of points on two neighboring lines of force with L = 1.25 are shown in Table 1.

TABLE 1

	R	θ	ϕ	B
Point A	1.114219	2.086164	5.140538	0.19732
B	1.159360	1.622298	5.134288	0.19732
C	1.093869	2.088672	5.335743	0.19732
D	1.151725	1.612954	5.276995	0.19732

Where R is in earth radii and θ, ϕ are in radians. We find that

$$\frac{u_d\ (A)}{u_d\ (B)} = \frac{AC}{BD} = \frac{.188208}{.165111} = 1.140$$

It is sometimes convenient to use $\dot{\phi}$ the angular drift velocity; projecting equation (7) along the direction $d\phi$ gives

$$\frac{\dot{\phi}\ (A)}{\dot{\phi}\ (B)} = \frac{\phi\ (C) - \phi\ (A)}{\phi\ (D) - \phi\ (B)} = \frac{.195205}{.142707} = 1.368$$

This checks with the results given by Newkirk and Walt (1964) in their plots of $\dot{\phi}$ against ϕ. The difference between $\dot{\phi}$ at the northern and southern ends of the line of force is due to two factors. One factor is the spread of the lines of force. The other factor is the change in $dx/d\phi$ where dx is a distance measured in the direction AC. Welch, et al, (1963) have a graph of the variation of mirror point height as ϕ is varied: the slope of this graph is approximately $dx/d\phi$. This graph shows that $dx/d\phi$ can differ appreciably from the northern to the southern mirror point.

A Canonical Coordinate System

Ray (1963) has discussed a coordinate system (α, β) such that

$$\bar{B} = \bar{\nabla}_\alpha \times \bar{\nabla}_\beta \tag{8}$$

and such that α and β are constant along a line of force. Northrop and Teller (1960) used this coordinate system in deriving an expression for the drift velocity. Ray (1963) showed that any function which is constant along a line of force may be chosen as α, and that β is then determined. If γ is any other quantity which is constant along a line of force then

$$\beta = \int \frac{B \; d\gamma}{\nabla_\alpha \times \nabla_\gamma} \tag{9}$$

where the path of integration in the direction $\bar{B} \times \bar{\nabla}_\alpha$. Consider the path of integration and let $d\chi$ denote an element of this path then

$$\beta = \int \frac{B \; d\gamma}{|\nabla_\alpha| \, d\gamma | d\chi} = \int \frac{B \; d\chi}{|\nabla_\alpha|} \tag{10a}$$

or

$$\frac{d\chi}{d\beta} = \frac{|\nabla\alpha|}{B} \tag{10b}$$

Let us now choose α to be equal to J for a fixed B_m. Then $d\chi$ denotes a distance in the direction $\bar{B} \times \nabla_\alpha$, that is it is in the direction of average drift. Using equation (2), and the fact that ∇J is perpendicular to $\bar{B}$,

$$\frac{d\chi}{d\beta} = \frac{e}{c} u_d \; T \tag{10c}$$

Since β is constant along a line of force, this equation is similar to the result derived in the preceeding section. $u_d T$ is proportional to $d\chi$ and $d\chi$ is the separation of the lines of force measured in the direction $B \times \nabla J$. It is possible to give a meaning to β; since the drift velocity is $d\chi/dt$, it follows that

$$\frac{d\beta}{dt} = \frac{c}{eT} \tag{11}$$

so that β is the time taken to drift from some reference line, measured in units of eT/c. The definition (10b) is exact; it does not contain any approximation. This analysis shows that there are an infinite number of α, β coordinate systems. Particles with different values of B_m lie on different shells; the above results show that there is a coordinate system appropriate to each value of B_m. On any shell, if B_m is taken to have the value $4.85M/I^3$, where M is the dipole magnetic moment of the earth, then I is equal to the L defined by McIlwain (1961). In the neighborhood of any shell we can choose the coordinates so that $\alpha = L$; however, this agreement does not hold for other shells since B_m must be kept constant.

The Drift Velocity and Calculation of an Average Atmosphere

Consider a particle with a certain value of B_m and a certain energy. It is known that the particle will, if scattering and loss processes are ignored, stay on a certain shell which is defined by B_m and by J, the integral invariant. Let $\rho(\bar{r})$ be some property of the atmosphere, for example ρ might be the concentration of oxygen at $\bar{r}$. The average of ρ as seen by a trapped particle is

$$\rho_{av} = S(\rho) \;/\; S\;(1) \tag{12a}$$

where

$$S(\rho) = \iint \frac{\rho\;(\bar{r})\;ds\;d\phi}{V_{//}\;\dot{\phi}} \tag{12b}$$

We have shown in the previous section that $d\;\phi \;/\; \dot{\phi}$ is independent of s, hence $S(\rho)$ is separable

$$S\;(\rho) \;=\; \int \frac{S\;(\rho,\;\phi)}{\dot{\phi}}\;d\phi \tag{13}$$

where

$$S\;(\rho,\phi) \;=\; \int \frac{\rho\;(\bar{r})}{V_{//}}\;ds_{\phi} \tag{14}$$

The subscript ϕ on ds denotes that integration is along the line of force with label ϕ. When evaluating (13), $d\;\phi \;/\; \dot{\phi}$ can be taken at any point on the line of force; a convenient point to use is the geomagnetic equator. The numerical calculation of $\bar{\rho}$ can be performed as follows: Choose many lines of force. Evaluate the integral of equation (14) for each line of force. Find the geomagnetic equator on each line of force and the angular drift velocity at that point. Finally, perform the integration (13).

Relation between Drift Velocities with Different B_m

We have shown that the average drift velocity at any point is related, in a simple geometrical fashion, to the drift velocity at the equator. It would therefore be advantageous to be able to compute u_d at the equator in a simple fashion. For particles which mirror at the equator we have from equation (1)

$$u_d = v_d = \frac{m \gamma c \varkappa v^2}{2 e B} \tag{15}$$

where $\varkappa$ is the curvature of the line of force.

The best way of computing $\varkappa$ is to use the relation given by Lew (1961). If $\bar{e}$ and $\bar{n}$ denote unit vectors along the tangent and normal, then

$$\frac{\overline{\mathrm{grad}\ B}}{B} = \frac{\bar{e}}{B} \cdot \frac{\partial B}{\partial s} + \varkappa \bar{n}$$

which, at the equator reduced to

$$\varkappa = \frac{|\mathrm{grad}\ B|}{B} \tag{16}$$

We can compare equation (2) and equation (13) for particles mirroring near the equator. For points near the equator let

$$B = B_o + a^2 s^2 \quad \text{where } a^2 = \frac{d^2 B}{2 d s^2} \tag{17}$$

and B_o is B at the equator. Let S_m denote the distance to the mirror point, then

$$B_m = B_o + a^2 S_m^2 \tag{18}$$

The bounce time for the particles is

$$T = \frac{4}{V} \int_o^{S_m} ds \bigg/ \left(1 - \frac{B}{B_m}\right)^{\frac{1}{2}}$$

putting $s = \sin\theta$ reduces the integral to a simple form and

$$T = \frac{2}{Va} \sqrt{B_m} \tag{19}$$

Applying the same transformation to J gives

$$J = 4p \int_0^{S_m} (1 - B/B_m)^{\frac{1}{2}} \, ds \tag{20a}$$

$$= \frac{\pi p}{a} \frac{B_m - B_o}{\sqrt{B_m}} = \frac{T\,pv}{2} \left(1 - \frac{B_o}{B_m} \right) \tag{20b}$$

hence differentiating with B_m fixed

$$\nabla J = \frac{-T\,p\,v}{2\,B_m} \nabla B_o + \frac{p\,v}{2\,B_m} (B_m - B_o) \nabla T \tag{21}$$

putting $B_m = B_o$ for particles at the equator,

$$\frac{c}{e\,B\,T} \nabla J = \frac{c}{eB} \frac{p\,v}{2\,B} \nabla B = \frac{-v\,p\,c\,\varkappa}{2\,e\,B}$$

which is identical with (15).

To compute the drift velocity at the equator, of particles which do not mirror at the equator, we can proceed as follows: McIlwain (1961) has shown that a certain quantity, which he calls L, is almost constant along a line of force, and that there is a relation between J and B, namely,

$$\frac{L^3 B}{M} = F \left(\frac{I^3 B}{M} \right) \tag{22}$$

where $I = J / 2p$, M is the dipole moment of the Earth and F is a function given by McIlwain.

Now at the equator

$$L = (M/B_o)^{1/3} \qquad (23)$$

The implication of equations (22) and (23) is that if we know B_o then we can find L, and assuming L is constant along the line of force, we can find I for any point on the line of force, thus I is the solution of the equation

$$\frac{B_m}{B_o} = F\left(\frac{I^3 B_m}{M}\right) \qquad (24)$$

To find ∇I we could proceed as follows: Find I for the equatorial value B_o, find I for equatorial value $B_o + h \nabla B$, then ∇I is the difference of the I's divided by h. Actually it is not necessary to do this calculation because the ratio of ∇I at one B_m value to the ratio of ∇I at another B_m value is almost independent of longitude. We can see this as follows. For a given B_{m1} take two shells with integral invariants I and $I + \delta I$. ∇I at any point is proportional to the separation of these two shells. Consider a line of force on the I shells, take a particle whose mirror point B value is equal to B_{m2}, evaluate its integral invariant and denote the result by I*. Take a similar line on the $I + \delta I$ shells, denote the result by $I^* + \delta I^*$. The ∇I for the second particle is proportional to the separation of the shells I* and $I^* + \delta I^*$. The shells I and I* coincide at one line of force, we assume that they will coincide at all lines of force; similarly, for the $I + \delta I$ and $I^* + \delta I^*$ shells. Thus ∇I for the second particle is proportional to the separation of I* and $I^* + \delta I^*$, which is the same as the separation of I and I + δ I, and this is proportional to ∇I for the first particle, thus

$$\nabla I(B_{m1}) \Big/ \nabla I(B_{m2}) = \text{a constant} \tag{25}$$

We have made the assumption that shells which coincide along one line of force will always coincide. We could have taken the alternative assumption that L is constant along a line of force; this actually leads to a similar result. For fixed B_m let us write (24) in the form

$$B_o = f(I) \tag{26a}$$

and let

$$B_o + h \nabla B_o = f(I + \delta I) = f(I) + \delta I \, f'(I) \tag{26b}$$

then

$$\nabla I = \frac{\delta I}{h} = \frac{\nabla B_o}{f'(I)} \tag{27}$$

Using McIlwain's equation (7) we find

$$\nabla I = \frac{-B_m \nabla B_o}{B_o \;\; G} \quad \frac{I}{(B_m - B_o)} \tag{28}$$

where

$$G = 3 \sum_{1}^{6} n \, a_n \, x^{n-1}$$

and a_n and x are defined by McIlwain. For points near the equator a_2 through a_6 are zero and a_1 = .333338. (We use the improved values of

McIlwain's coefficients as given the report by Dudziak et al. (1963); hence using equation (20)

$$\nabla J = 2p \; \nabla I = \frac{- B_m \nabla B_o}{B_o} \quad \frac{T p v}{2 B_m} \tag{29}$$

Equation (25) checks with equation (19) at the equator: equation (28) should be valid for all B_m. Collecting together equations (2), (28), (15) and (16), the drift at the equator is

$$u_d(B_m) = \frac{4 \; I \; B_m}{(B_m - B_o) \; GTv} \quad u_d(B_o) \tag{30}$$

where $u_d(B)$ refers to particles mirroring at the point B.

For many calculations we are only interested in the variation of u with ϕ where I, v and B_m are fixed: from equation (30) considered as a function of ϕ

$$u_d(B_m) = \text{a constant times} \; \frac{u_d(B_o)}{(B_m - B_o) \; T} \tag{31}$$

where T is the bounce period. For particles which mirror near the equator, equation (20) shows that $T(B_m - B_o)$ is constant; for higher values of B_m it can be shown that $T(B_m - B_o)$ will vary with longitude.

Figure 2 should appear near here

The results are illustrated in Figure 2. One curve represents ϕ for a particle mirroring at the geomagnetic equator; the particular equator considered here has B = .1596 gauss, that is L = 1.25. This first curve was computed from equations (15) and (16) and has been normalized so that its average value is unity.

The second curve represents the ratio

$$f(\phi) = u_d(B_m)\,G\,T / u_d(B_o)\,I$$

normalized to an average value of unity. $u_d(B_m)$ was evaluated at the geomagnetic equator using the equation (2): B_m was taken as .22 gauss (corresponding to a minimum height of 250 km) and B_o was .1596 gauss. Since B_o and B_m were held fixed, I varies by about 2% as ϕ was varied. According to equation (30), the ratio $f(\phi)$ should be constant. It can be seen that f varies by less than 2%. For this value of L and B_m, the drift at the mirror point varies from .8 to 1.2 times its average value.

A similar calculation for the L = 1.9, with B_m again equal to .22, gave $f(\phi)$ varying from .986 to 1.016 times its average value.

Conclusion

There are three conclusions. u_d the average drift velocity varies along a line of force, the variation is proportional to the spatial separation of lines in the same J shell. u_d at the geomagnetic equator varies when B_m, the mirror point B, is varied; however, the ratio of u_d's for two values of B_m is almost independent of changes in longitude. These relations among the u_d's make it possible to compute u_d at any point by a relatively simple calculation; they also considerably simplify the calculation of average atmospheres.

Acknowledgements

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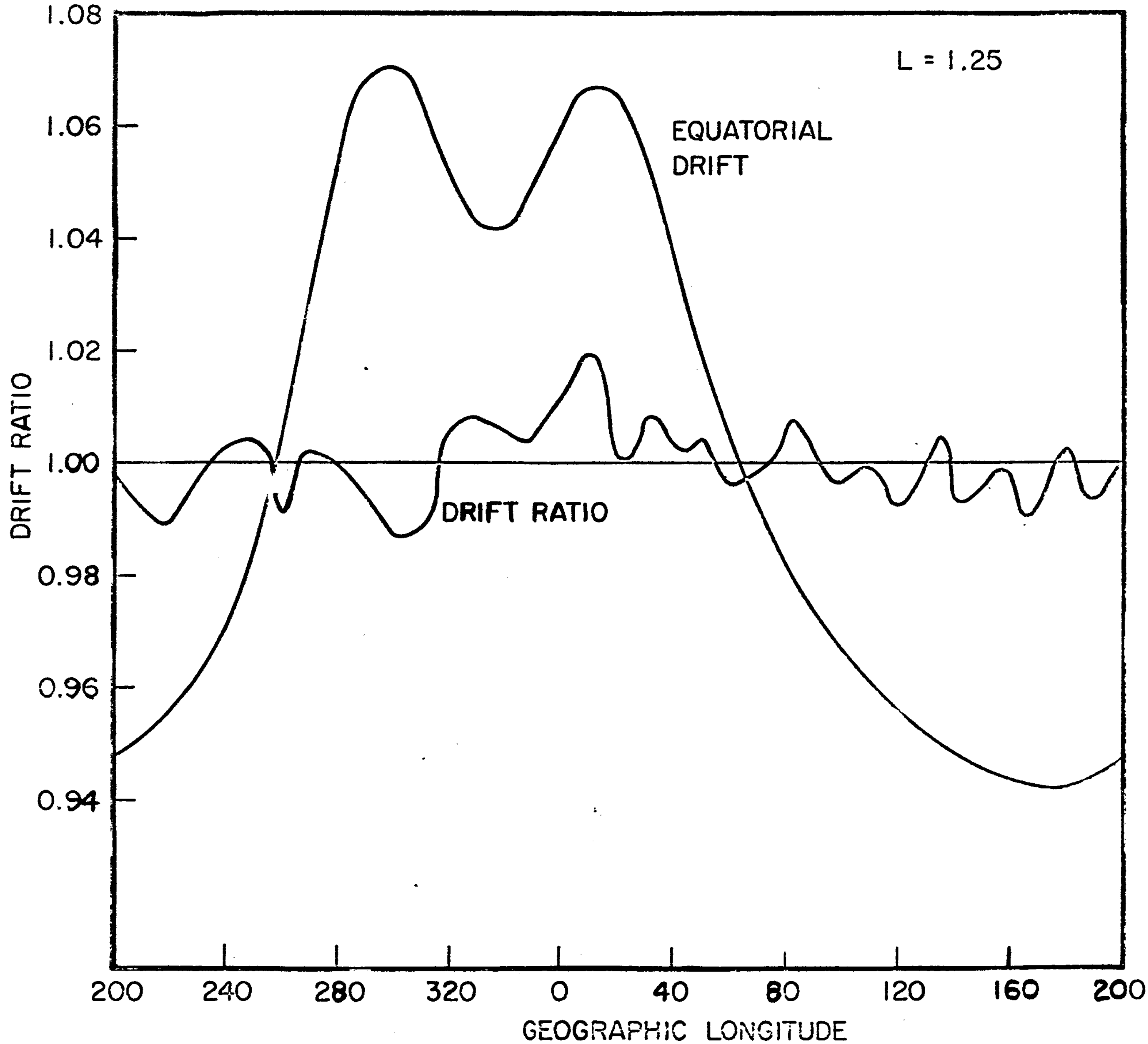
Figure Captions

Figure 1. Two neighboring lines of force AB, CD and the associated particle paths AB' and CD', (Page 16).

Figure 2. Angular drift at the equator for a particle with l = 1.25, compared with the ratio of (drift at the equator of particle with $B_m = .22$) / (drift of particle with $B_m = B_o$). Both curves are normalized to an average value of unity.


1.08
1.06
1.04
1.02
1.00
0.98
0.96
0.94
DRIFT RATIO
L = 1.25
EQUATORIAL
DRIFT
DRIFT RATIO
200
240
280
320
0
40
80
120
160
200
GEOGRAPHIC LONGITUDE

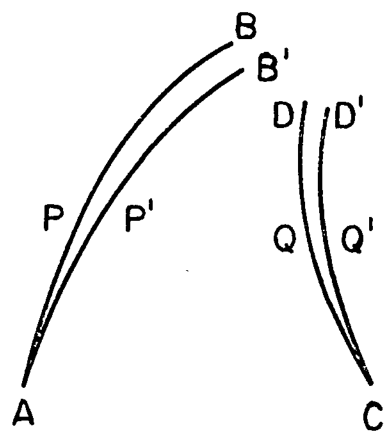


Figure 1.